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Globale Stability in a Viral Infection Model with Beddington-DeAngelis Functional Response

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Abstract: The stability of a mathematical model for viral infection with Beddington-DeAngelis functional response is considered in this paper. If the basic reproduction number $R_0 \le 1$, by the Routh-Hurwitz criterion and Lyapunov function, the uninfected equilibrium E_0 is globally asymptotically stable. Then, the global stability of the infected equilibrium E_1 is obtained by the method of Lyapunov function.

Keywords: Beddington-DeAngelis, equilibrium, Global dynamics, HIV model.

1. INTRODUCTION

Human Immunodeficiency Virus and Acquired Immune Deficiency Syndrome (AIDS) have received much attention from the first case of AIDS was diagnosed on December 1st in 1981. It is proven to be valuable in understanding the population dynamics of viral load in vivo with mathematical models. In the last decade, many mathematical models have been developed to describe the infection with Human Immunodeficiency virus (HIV) (see[1-9]). Nowak *et al*. [1, 3] proposed the following model:

$$\begin{cases} \dfrac{dx}{dt} = \lambda - dx - \beta xv, \\ \dfrac{dy}{dt} = \beta xv - ay, \\ \dfrac{dv}{dt} = ky - rv. \end{cases} \tag{1.1}$$

where $x(t), y(t)$ and $v(t)$ represent the numbers(densities) of healthy $CD4^+$ T-cells, infected $CD4^+$ T-cells and viral particles at time t respectively. The model assumed that healthy $CD4^+$ T-cells are infected at a rate βxv, infected $CD4^+$ T-cells are lost at a rate ay, and virus are produced by infected $CD4^+$ T-cells at a rate ky and removed at a rate rv. In model (1.1), it is also assumed that healthy $CD4^+$ T-cells are input at a constant rate λ, and die at a rate dx.

Although the infection rate is bilinear in most HIV-I models with the virus v and healthy $CD4^+$ T- cells x, actual incidence rates are probably not linear over the entire range of v and x. Thus, it is reasonable for our paper to assume that the infection rate of the form $\frac{\beta xv}{1+mx+nv}$, where $m, n \ge 0$ are constants. The function response $\frac{\beta xv}{1+mx+nv}$ was introduced by Beddington [10] and DeAngelis *et al*. [11].

In this paper, we consider a HIV-I model with Beddington-DeAngelis function response as follows:

$$\begin{cases} x' = \lambda - dx - \dfrac{\beta xz}{1+mx+nz}, \\ y' = \dfrac{\beta xz}{1+mx+nz} - ay, \\ z' = ky - bz. \end{cases} \tag{1.2}$$

The biological meanings of these parameters are the similar to those appearing parameters in model (1.1).

2. EQUILIBRIA AND GLOBAL STABILITY ANALYSIS

Let

$$R_0 = \frac{\beta \lambda k}{(d + m\lambda)ba}$$

Then R_0 is the basic reproductive number of model (1.2), which describes the average number of newly infected T-cells generated from one infected T-cells. We can obtain that $E_0 = (x_0, 0, 0)$ is an uninfected equilibrium where $x_0 = \lambda / d$, and $E_1 = (x_1, y_1, z_1)$ is a infected equilibrium of model (1.2) if and only if $R_0 > 1$ where

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$x_1 = \frac{ba + nk\lambda}{\beta k + nkd - bam}$, $y_1 = \frac{\lambda\beta k - \lambda bam - dba}{a(\beta k + nkd - bam)}$, $z_1 = \frac{k(\lambda\beta k - \lambda bam - dba)}{ba(\beta k + nkd - bam)}$

Now, we begin to study the stabilities of these two equilibria.

Firstly, we begin to study the stability of the uninfected equilibrium $E_0 = (x_0, 0, 0)$. Evaluating the Jacobian matrix of model (1.2) at E_0 gives

$$J(E_0) = \begin{bmatrix} -d & 0 & -\frac{\beta\lambda}{d + m\lambda} \\ 0 & -a & \frac{\beta\lambda}{d + m\lambda} \\ 0 & k & -b \end{bmatrix}.$$

By simple computations, the characteristic equation is

$$\Delta(U) = U^3 + (a + d + b)U^2 + (ad + bd + ab - \frac{k\beta\lambda}{d + m\lambda})U + adb - \frac{k\beta\lambda}{d + m\lambda}d$$
$$= U^3 + A_1U^2 + A_2U + A_3$$

Where $A_1 = a + d + b > 0$, $A_2 = ad + bd + ab - \frac{k\beta\lambda}{d + m\lambda} > 0$, $A_3 = adb - \frac{k\beta\lambda}{d + m\lambda}d > 0$, and

$$A_1A_2 - A_3 = a^2(b + d) + b^2(a + d) + d^2(a + b) + 2abd - \frac{k\beta\lambda}{d + m\lambda}(a + b) > 0$$

if $R_0 < 1$. According to the Routh-Hurwitz criterion, it is obtained that uninfected equilibrium E_0 is locally asymptotically stable when $R_0 < 1$.

When $R_0 > 1$ it is easy to obtain that $A_3 < 0$, and $\Delta(0) = A_3 < 0, \Delta(+\infty) = +\infty$. That is to say the charac-terristic equation has positive solution. So E_0 is unstable when $R_0 > 1$.

Moreover, we construct a Lyapunov function for studying the global stability. Let

$$L_0(x, y, z) = \frac{1}{1 + mx_0}(x - x_0 - x_0 \ln\frac{x}{x_0}) + y + \frac{a}{k}z$$

Along the trajectories of model (1.2),

Because $2 - x_0/x - x/x_0 \le 0$, if $R_0 \le 1$ it can be obtained $\frac{dL_0(x, y, z)}{dt} \le 0$ for all $x > 0, y > 0, z > 0$. Furthermore, when and only when $x = x_0, y = 0, z = 0$, we have $\frac{dL_0}{dt}\Big|_{(x_0,0,0)} = 0$. By the LaSalle invariance principle, when $R_0 \le 1$ the uninfected equilibrium E_0 is globally asymptotically stable.

Summarizing the discussion above, we obtained the following conclusion.

Theorem 2.1 The uninfected equilibrium E_0 is globally asymptotically stable when $R_0 \le 1$ and is unstable when $R_0 > 1$.

Then we begin to analysis the stability of infected equilibrium E_1. Let

$$L_1(x, y, z) = x - x_1 - \int_{x_1}^{x} \frac{ay_1}{\beta z_1 t / (1 + mt + nz_1)}dt + y - y_1 - y_1 \ln\frac{y}{y_1} + \frac{a}{k}(z - z_1 - z_1 \ln\frac{z}{z_1})$$

Along the trajectories of model (1.2), we obtained

$$L_1'|_{(1.2)} = x - x_1 - \int_{x_1}^{x} \frac{ay_1}{\beta z_1 t / (1 + mt + nz_1)}dt + y - y_1 - y_1 \ln\frac{y}{y_1} + \frac{a}{k}(z - z_1 - z_1 \ln\frac{z}{z_1})$$
$$= x' - \frac{ay_1(1 + mx + nz_1)}{\beta z_1 x}x' + y' - \frac{y_1}{y}y' + \frac{a}{k}(z' - \frac{z_1}{z}z')$$
$$= \lambda - dx - \frac{ay_1(1 + mx + nz_1)}{\beta z_1 x}(\lambda - dx - \frac{\beta zx}{1 + mx + nz})$$
$$- \frac{y_1}{y}\frac{\beta zx}{1 + mx + nz} + ay_1 - \frac{abz}{k} - \frac{z_1 ay}{z} + \frac{abz_1}{k}$$

Since (x_1, y_1, z_1) is the equilibrium point of (1.2), we have

$$\lambda = dx_1 + ay_1, \frac{y_1}{z_1} = \frac{b}{k}, ay_1 = \frac{\beta x_1 z_1}{1 + mx_1 + nz_1}$$

So

$$L_0'|_{(1.2)} = \frac{1}{1 + mx_0}(x' - x_0\frac{1}{x}x') + y' + \frac{a}{k}z'$$
$$= \frac{1}{1 + mx_0}(1 - x_0\frac{1}{x})(\lambda - dx - \frac{\beta xz}{1 + mx + nz}) + \frac{\beta xz}{1 + mx + nz} - ay + \frac{a}{k}(ky - bz)$$
$$= \frac{1}{1 + mx_0}(\lambda - dx - \frac{x_0}{x}\lambda + x_0 d) + \frac{1}{1 + mx + nz}(\frac{\beta x_0 z}{1 + mx_0} - \frac{\beta xz}{1 + mx_0}) - \frac{ab}{k}z + \frac{\beta xz}{1 + mx + nz}$$
$$= \frac{x_0 d}{1 + mx_0}(2 - \frac{x_0}{x} - \frac{x}{x_0}) + \frac{1}{1 + mx + nz}(\frac{\beta x_0 z}{1 + mx_0} - \frac{\beta xz}{1 + mx_0} - \frac{abz(1 + mx + nz)}{k} + \beta xz)$$
$$= \frac{x_0 d}{1 + mx_0}(2 - \frac{x_0}{x} - \frac{x}{x_0}) + \frac{1}{1 + mx + nz}(\frac{mx\beta x_0 z + \beta x_0 z}{1 + mx_0} - \frac{abz(1 + mx + nz)}{k})$$
$$= \frac{x_0 d}{1 + mx_0}(2 - \frac{x_0}{x} - \frac{x}{x_0}) + \frac{1}{1 + mx + nz}(\frac{\beta\lambda z(mx + 1)}{d + m\lambda} - \frac{abz(1 + mx)}{k} - \frac{abnz^2}{k})$$
$$= \frac{x_0 d}{1 + mx_0}(2 - \frac{x_0}{x} - \frac{x}{x_0}) + \frac{abz(1 + mx)}{(1 + mx + nz)k}(R_0 - 1) - \frac{abnz^2}{k(1 + mx + nz)}$$

Moreover

$$ay_1(1+\frac{z}{z_1}-\frac{z}{z_1}\frac{1+mx+nz_1}{1+mx+nz}-\frac{1+mx+nz}{1+mx+nz_1})$$
$$=ay_1(\frac{nz_1-nz}{1+mx+nz_1}+\frac{z}{z_1}\frac{nz-nz_1}{1+mx+nz})$$
$$=ay_1n(z_1-z)(\frac{1}{1+mx+nz_1}-\frac{z}{z_1}\frac{1}{1+mx+nz})$$
$$=ay_1n(z_1-z)\frac{z_1-z+mx(z_1-z)}{z_1(1+mx+nz)(1+mx+nz_1)}$$
$$=\frac{ay_1n(z_1-z)^2(1+mx)}{z_1(1+mx+nz)(1+mx+nz_1)}\geq 0$$

and

$$4-\frac{x_1}{x}\frac{1+mx+nz_1}{1+mx_1+nz_1}-\frac{xzy_1}{x_1z_1y_1}\frac{1+mx_1+nz_1}{1+mx+nz}-\frac{z_1y}{zy_1}-\frac{1+mx+nz}{1+mx+nz_1}\leq 0.$$

We got for all $x>0, y>0, z>0$, $\frac{dL_1(x,y,z)}{dt}\leq 0$ holds. Moreover, $\frac{dL_1}{dt}=0$ when and only when $x=x_1$, $y=y_1$, $z=z_1$. According to LaSalle invariance principle, the infected equilibrium E_1 is globally asymptotically stable.

Theorem 2.2 The infected equilibrium E_1 exists and is globally asymptotically stable when $R_0>1$.

CONCLUSION

In this paper, we have investigated a HIV-I mathematical model with Beddington-DeAngelis function response. According to the Routh-Hurwitz criterion and LaSalle invariance principle, we obtained the following conclusion: 1) when $R_0\leq 1$ the uninfected equilibrium E_0 is globally asymptotically stable; 2) when $R_0>1$ the infected equilibrium E_1 is globally asymptotically stable.

CONFLICT OF INTEREST

The authors confirm that this article content has no conflict of interest.

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REFERENCES

[1] Nowak MA, Bangham CRM. Population dynamics of immune responses to persistent virus. Science 1996; 272(5258): 74-9.
[2] Perelson A, Nelson P. Mathematical models of HIV dynamics *in vivo*. SIAM Rev 1999; 41: 3-44.
[3] Nowak MA, May RM. Virus Dynamics. New York: Oxford University press, 2000: 15-80.
[4] Yang Y, Xiao Y. Threshold dynamics for an HIV model in periodic environments. J Math Anal Appl 2010; 361(1): 59-68.
[5] Wang X, Wang W. An HIV infection model based on a vectored immunoprophylaxis experiment. J Theoret Biol 2012; 313: 127-35.
[6] Xu R. Global stability of a delayed epidemic model with latent period and vaccination strategy. Appl Mathemat Mode 2012; 36(11): 5293-300.
[7] Wang X, Elaiw A, Song X. Global properties of a delayed HIV infection model with CTL immune response. Appl Mathemat Comput 2012; 218(18): 9405-14.
[8] Wang X, Tao Y. Lyapunov function and global properties of virus dynamics with CTL immune response. Int J Biomathemat 2008; 1(4): 443-8.
[9] Wang X,Tao Y, Song X. A delayed HIV-I infection model with Bedington-DeAngelis functional response. Nonlinear Dyn 2010; 62(1-2): 67-72.
[10] Beddington JR. Mutual interference between parasites or predators and its effect on searching efficiency. J Animal Ecol 1975; 44(1): 331-40.
[11] DeAngelis DL, Goldstein RA, O'Neill RV. A model for trophic interaction. Ecology 1975; 56(4): 881-92.

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